

TANGENT CONES AND C^1 REGULARITY OF DEFINABLE SETS

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ABSTRACT. Let $X \subset \mathbb{R}^n$ be a connected locally closed definable set in an o-minimal structure. We prove that the following three statements are equivalent: (i) X is a C^1 manifold, (ii) the tangent cone and the paratangent cone of X coincide at every point in X , (iii) for every $x \in X$, the tangent cone of X at the point x is a k -dimensional linear subspace of \mathbb{R}^n (k does not depend on x) varies continuously in x , and the density $\theta(X, x) < 3/2$.

1. INTRODUCTION

Let X be a subset of \mathbb{R}^n and let $x \in \mathbb{R}^n$. The *tangent cone* $\text{tg}_x X$ and *paratangent cone* $\text{ptg}_x X$ of X at the point x are defined as follows: if $x \notin \overline{X}$, $\text{tg}_x X = \text{ptg}_x X = \{0\}$, and otherwise,

$$\text{tg}_x X := \{au \mid a \in \mathbb{R}, a \geq 0, u = \lim_{i \rightarrow \infty} \frac{x_i - x}{\|x_i - x\|}, \{x_i\} \subset X, \{x_i\} \rightarrow x\},$$

$$\text{ptg}_x X := \{au \mid a \geq 0, u = \lim_{i \rightarrow \infty} \frac{x_i - y_i}{\|x_i - y_i\|}, X \supset \{x_i\} \rightarrow x, X \supset \{y_i\} \rightarrow x\}.$$

Note that $\text{tg}_x X$ and $\text{ptg}_x X$ are closed sets in \mathbb{R}^n . We denote by $\text{tg} X := \{(x, v), x \in X, v \in \text{tg}_x X\}$ and $\text{ptg} X := \{(x, v), x \in X, v \in \text{ptg}_x X\}$.

Characterizing C^1 submanifolds of \mathbb{R}^n in terms of their tangent cones has been studied by many authors, see for example [8], [10], [2], [6], or a survey of Bigolin and Golo [1]. In this paper we restrict ourselves to this problem in the context of o-minimal structures. We first prove that a connected locally closed definable subset of \mathbb{R}^n is a C^1 manifold if and only if its tangent cone and paratangent cone coincide at every point (Theorem 3.7). This result is a strong version of the two-cones coincidence theorem (Theorem 3.6) which was initially proved by Tierno [10]. The result is no longer true if definability is omitted (Remark 3.9).

Next, we discuss a result recently established by Ghomi and Howard [6] that if $X \subset \mathbb{R}^n$ is a locally closed set such that for each $x \in X$, $\text{tg}_x X$ is a hyperplane (i.e., a $(n-1)$ linear subspace of \mathbb{R}^n), and varies continuously in x , then X is a union of C^1 hypersurfaces. Moreover, if the lower density $\Theta(X, x)$ is at most $m < \frac{3}{2}$ for every $x \in X$ then X is a C^1 hypersurface. A natural question thus arises here is whether the result remains true if $\text{tg}_x X$ are k -planes with $k < n-1$.

In section 4, we show in Example 4.3 that the first statement in the result of Ghomi-Howard is not always true if $k < (n-1)$. We also prove that the second statement is still valid, more precisely that if X is a locally closed definable set such that for every $x \in X$, $\text{tg}_x X$ is a k -dimensional linear subspace (k is independent of x) varying continuously in x and the density $\theta(X, x) < 3/2$ (need not be upper

bounded by an $m < \frac{3}{2}$), then X is a C^1 manifold (Theorem 4.7). Notice that, in general, notions of lower density $\Theta(X, x)$ and density $\theta(X, x)$ are different. Nevertheless, with the conditions on tangent cones as above, they coincide. Therefore, our result can be considered as a generalization of the result of Ghomi-Howard.

Throughout the paper, \mathbb{R}^n denotes the n -dimensional Euclidean space equipped with the standard norm $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$ where $x = (x_1, \dots, x_n)$; $\overline{\mathbf{B}}^n(x, r)$, $\mathbf{B}^n(x, r)$ and $\mathbf{S}^{n-1}(x, r)$ denote respectively the closed ball, the open ball and the sphere in \mathbb{R}^n of radius r centered at x . Let X be a subset in \mathbb{R}^n . Denote by \overline{X} the closure of X in \mathbb{R}^n and by $\partial X := \overline{X} \setminus X$ the boundary of X . Let f be a map. We denote by Γ_f the graph of f .

The Grassmanian \mathbb{G}_n^k of all k -dimensional linear subspaces of \mathbb{R}^n is endowed with the metric δ defined as follows: for P, Q in \mathbb{G}_n^k ,

$$\delta(P, Q) = \sup_{v \in P, \|v\|=1} \{\|v - \pi_Q(v)\|\},$$

where $\pi_Q : \mathbb{R}^n \rightarrow Q$ is the orthogonal projection from \mathbb{R}^n to Q . Following [6], we say that $P \in \mathbb{G}_n^k$ and $Q \in \mathbb{G}_n^k$ are *orthogonal* when $\delta(P, Q) = 1$. Remark that this terminology does not coincide with the usual notion of orthogonality for general subspaces in Euclidean geometry: $P = \{(x, y, z); x = 0\}$ and $Q = \{(x, y, z); y = 0\}$ are orthogonal according to our definition but not all vectors in P are orthogonal to any vector in Q .

By a *k -dimensional C^1 manifold in \mathbb{R}^n* (or C^1 manifold for simplicity) we mean a subset of \mathbb{R}^n , locally C^1 diffeomorphic to \mathbb{R}^k ; a *hypersurface* in \mathbb{R}^n is a C^1 manifold in \mathbb{R}^n of dimension $n - 1$.

Let $X \subset \mathbb{R}^n$. In the paper, we often denote by $\pi_x : \mathbb{R}^n \rightarrow \text{tg}_x X$ the orthogonal projection. By abuse of notation, here we identify $\text{tg}_x X$ with its translation $\{x + \text{tg}_x X\}$.

By a *definable set* we mean a set which is definable (with parameters) in an o-minimal expansion $(\mathbb{R}, <, +, \cdot, \dots)$ of the ordered field of real numbers. Definable sets form a large class of subsets of \mathbb{R}^n : for instance, any semi-algebraic set, any sub-analytic set is definable. We refer the reader to [11], [5] for the basic properties of o-minimal structures. In the paper, we will use Curve selection Lemma ([5], Theorem 3.2), Uniform finiteness on fibers ([5] Theorem 2.9) and Hardt's definable triviality Theorem ([5], Theorem 5.22) without repeating the references.

2. BUNDLE OF VECTOR SPACES

Let X be a subset of \mathbb{R}^n . Let $E \subset X \times \mathbb{R}^n$. For $x \in X$ we denote by

$$E_x := \{v \in \mathbb{R}^n : (x, v) \in E\}$$

the *fiber* of E at the point x . For $U \subset X$, we set

$$E|_U := \{(x, v) \in E : x \in U\}$$

and call it *the restriction of E to U* . If every fiber of E is a linear subspace of \mathbb{R}^n we call E a *bundle of vector spaces over X* , or a *bundle over X* , or just a *bundle* if the base X is clear from the context. We call E a *trivial bundle* if all its fibers have the same dimension, and a *closed bundle* if it is a closed set in \mathbb{R}^{2n} .

Suppose E is a trivial bundle over X . If the map $X \rightarrow \mathbb{G}_n^k$ defined by $x \mapsto E_x$ is continuous, we say that E is a *continuous trivial bundle*.

Lemma 2.1 ([7], Propositions I, II, pages 39, 40). *Let $E \subset X \times \mathbb{R}^n$.*

- (i) *If E is a closed trivial bundle then E is continuous.*
- (ii) *If E is a closed bundle then the function $x \mapsto \dim E_x$ is upper-semicontinuous, i.e., for $x \in X$ there is an open neighborhood U_x of x in X such that $\dim E_x \geq \dim E_y$ for every $y \in U_x$.*

Proof. (i) - Suppose that k is the dimension of fibers of E . If E is not continuous, there exists a sequence $\{x_i\} \subset X$ tending to x , $\lim_{i \rightarrow \infty} E_{x_i} \rightarrow \tau \in \mathbb{G}_n^k$ and $\tau \not\subset E_x$. By the closedness of E , $\tau \subset E_x$, which is a contradiction.

(ii) - Suppose the assertion is not true, i.e., there exist a point $x \in X$ and a sequence $\{x_m\}$ in X tending to x such that $\dim E_{x_m} > \dim E_x$. We may assume that $\lim_{m \rightarrow \infty} E_{x_m} = P \in \mathbb{G}_n^k$, since \mathbb{G}_n^k is compact. Note that $k > \dim E_x$. Since E is closed, $P \subset E_x$. This implies $k \leq \dim E_x$, which is a contradiction. \square

Lemma 2.2. *Let X be a locally closed subset of \mathbb{R}^n . If $\text{ptg}X$ is a trivial bundle then $\text{ptg}X$ is continuous.*

Proof. It follows directly from the definition of the paratangent cone that $\text{ptg}(\overline{X})$ is a closed set in \mathbb{R}^{2n} and $\text{ptg}_x X = \text{ptg}_x \overline{X}$ for every $x \in X$. If $V \subset X$ is a closed set in \mathbb{R}^n then $\text{ptg}X|_V = \text{ptg}\overline{X}|_V$ is a closed set in \mathbb{R}^{2n} .

Let $x \in X$. Since X is locally closed, there is W_x , a neighborhood of x in X , which is closed in \mathbb{R}^n . The restriction $\text{ptg}X|_{W_x}$ is then a closed set in \mathbb{R}^{2n} . Since $\text{ptg}X$ is a trivial bundle, so is $\text{ptg}X|_{W_x}$. By (i) in Lemma 2.1 $\text{ptg}X|_{W_x}$ is continuous, $\text{ptg}X$ is, therefore, a continuous trivial bundle. \square

3. TWO-CONES COINCIDENCE THEOREM

The aim of this section is to prove Theorem 3.7, a strong version of two-cones coincidence theorem of Tierno for definable sets.

We need the following two lemmas which generalize Lemma 3.3 and Lemma 3.1 in [6].

Lemma 3.1. *Let X be a locally closed subset of \mathbb{R}^n such that $\text{tg}X$ is a continuous trivial bundle of k -dimensional vector spaces. Let $x \in X$ and H be a k -plane in \mathbb{R}^n which is not orthogonal to $\text{tg}_x X$. Let $\pi : \mathbb{R}^n \rightarrow H$ the orthogonal projection. Then, there exists an open set U of x in X such that $\pi|_U : U \rightarrow H$ is an open map.*

Proof. The proof follows closely the proof of Lemma 3.3, [6].

By the continuity of $\text{tg}X$, we can choose an open neighborhood U of x in X such that for all $q \in U$, $\text{tg}_q X$ is not orthogonal to H , or equivalently that $\text{tg}_q X$ is transverse to H^\perp , the orthogonal complement of H in \mathbb{R}^n . We will prove that $\pi|_U$ is an open map. Fix $q \in U$. By the local closedness of X , there is an $r > 0$ small enough such that $W := X \cap \overline{\mathbf{B}}^n(q, r) \subset U$ is a compact set. Moreover, the boundary ∂W is in $\partial \overline{\mathbf{B}}^n(q, r)$, meaning $q \notin \partial W$. With r sufficiently small, we may assume that

$$\pi(q) \notin \pi(\partial W)$$

because, otherwise, there exists a sequence of positive numbers $\{r_i\}$ tending to 0 such that for each i , there is a point $q_i \in X \cap \partial \overline{\mathbf{B}}^n(q, r_i)$ with $\pi(q_i) = \pi(q)$. This implies that $\text{span}(q - q_i) \subset H^\perp$. As $i \rightarrow \infty$ we have $q_i \rightarrow q$ and the sequence $\text{span}(q - q_i)$ (extracting a subsequence if necessary) tends to a line $l \in \text{tg}_x X$. Thus, $l \in H^\perp \cap \text{tg}_q X$; but $\dim(H^\perp \cap \text{tg}_q X) = \dim H^\perp + \dim \text{tg}_q X - n = 0$ since H^\perp is transverse to $\text{tg}_q X$, which is a contradiction.

Since $\pi(\partial W)$ is a compact set, there is $s > 0$ such that

$$(3.1) \quad \overline{\mathbf{B}}^k(\pi(q), s) \cap \pi(\partial W) = \emptyset.$$

It suffices to show that $\pi(W)$ contains an open neighborhood of $\pi(q)$ in H .

Suppose on the contrary that $\overline{\mathbf{B}}^k(\pi(q), \varepsilon) \not\subset \pi(W)$, $\forall \varepsilon > 0$. Choose a point $q' \in \overline{\mathbf{B}}^k(\pi(q), s/2) \setminus \pi(W)$ and let s' be the distance from q' to $\pi(W)$. Note that $s' \leq s/2$. Since $\pi(W)$ is compact, $\overline{\mathbf{B}}^k(q', s') \cap \pi(W) \neq \emptyset$ and $\mathbf{B}^k(q', s') \cap \pi(W) = \emptyset$. For every $p \in \overline{\mathbf{B}}^k(q', s')$,

$$\|p - \pi(q)\| \leq \|p - q'\| + \|q' - \pi(q)\| \leq s' + s/2 \leq s.$$

This means that $\overline{\mathbf{B}}^k(q', s') \subset \overline{\mathbf{B}}^k(\pi(q), s)$. By (3.1), $\overline{\mathbf{B}}^k(q', s') \cap \pi(\partial W) = \emptyset$. Take $y' \in \overline{\mathbf{B}}^k(q', s') \cap \pi(W)$ and $y \in \pi^{-1}(y') \cap W$. Note that $y \notin \partial W$, so y is an interior point of W , and hence $\text{tg}_y W = \text{tg}_y X$ which is a linear subspace.

Since $\mathbf{B}^k(q', s') \cap \pi(W) = \emptyset$, no point of W is contained in the cylinder $C := \pi^{-1}(\mathbf{B}^k(q', s'))$. This implies that $\text{tg}_y X \subset \text{tg}_y \partial C$. Both $\text{tg}_y X$ and H^\perp are included in $\text{tg}_y \partial C$, so

$$\dim(\text{tg}_y X \cap H^\perp) \geq \dim \text{tg}_y X + \dim H^\perp - \dim \text{tg}_y \partial C = k + (n - k) - (n - 1) = 1.$$

This shows that $\text{tg}_y X$ is orthogonal to H , which is a contradiction. \square

Lemma 3.2. *Let $U \subset \mathbb{R}^k$ be an open set and $f : U \rightarrow \mathbb{R}^{n-k}$ be a map. Suppose that Γ_f is locally closed. If $\text{tg} \Gamma_f$ is a continuous trivial bundle of k -dimensional vector spaces and $\text{tg}_{(x, f(x))} \Gamma_f$ is not orthogonal to \mathbb{R}^k for every $x \in U$, then f is C^1 .*

Proof. We first prove that f is continuous. Suppose on the contrary that f is not continuous, meaning that there are $x \in U$ and a sequence $\{x_i\}$ in U tending to x such that $\lim_{i \rightarrow \infty} f(x_i) = y \neq f(x)$.

Since Γ_f and the orthogonal projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ satisfy the hypothesis of Lemma 3.1, there is an open set V of $(x, f(x))$ in Γ_f such that $\pi|_V$ is an open map. Set $W := \pi(V)$, which is an open neighborhood of x in \mathbb{R}^k . Since $\{x_i\}$ tends to x , there is $N \in \mathbb{N}$ such that $x_i \in W$ for all $i > N$. This implies that $(x_i, f(x_i)) \in V$ for all $i > N$.

If $y \neq \infty$, shrinking V so that $(x, y) \notin \overline{V}$, there is a neighborhood K of (x, y) in \mathbb{R}^n such that $K \cap V = \emptyset$. Since $f(x_i)$ tends to y , we have $(x_i, f(x_i)) \in K$ for all $i > N$ when N is large enough. This shows that $K \cap V \neq \emptyset$, which is a contradiction.

If $y = \infty$, $(x_i, f(x_i)) \notin V$ for all $i > N$ when N is large enough. This again gives a contradiction.

We now show that f is a C^1 map.

Let $\{a_1, \dots, a_k, b_1, \dots, b_{n-k}\}$ be the canonical basis of \mathbb{R}^n . For $x \in U$, consider the function $f_x^i(t) := f(x + ta_i)$. The graph $\Gamma_{f_x^i}$ of f_x^i is the intersection $\Gamma_{f_x^i} = \Gamma_f \cap (x, 0) + \text{span}(a_i, b_1, \dots, b_k)$. This implies that

$$\text{tg}_{(x, f(x))} \Gamma_{f_x^i} \subset \text{tg}_{(x, f(x))} \Gamma_f \cap \text{span}(a_i, b_1, \dots, b_{n-k}) =: l_x.$$

But l_x is a line, because $\text{tg}_{(x, f(x))} \Gamma_f$ is not orthogonal to $\mathbb{R}^k \times \{0\}^{n-k}$. On the other hand, since f is continuous, $\Gamma_{f_x^i}$ is a continuous curve, so $\text{tg}_{(x, f(x))} \Gamma_{f_x^i}$ has dimension at least 1. Then $\text{tg}_{(x, f(x))} \Gamma_{f_x^i} = l_x$, so f_x^i is differentiable at $t = 0$. Thus, f has partial derivatives at any point.

The bundle $\text{tg}_{(x, f(x))} \Gamma_f$ is continuous, hence l_x , its transverse intersection with $\text{span}(a_i, b_1, \dots, b_{n-k})$ is continuous. Therefore f has continuous partial derivatives on U , so f is C^1 . □

Remark 3.3. The statement of Lemma 3.1 [6] is similar to the statement of Lemma 3.2 except the local closedness of Γ_f is missing. This is a gap because f might not be continuous if Γ_f is not locally closed. For example, consider the function $f(x) = 0$ if x is a rational number, and $f(x) = 1$ otherwise. The tangent cone to Γ_f is the x -axis at any point, hence Γ_f is a continuous trivial bundle, but f is not continuous.

Theorem 3.4. *A locally closed set $X \subset \mathbb{R}^n$ is a C^1 manifold if and only if $\text{tg}X$ is a continuous trivial bundle and the restriction of the map $\pi_x : X \rightarrow \text{tg}_x X$ to some neighborhood of x in X is injective.*

Proof. The necessity is a trivial fact. We now prove the sufficiency. For $x \in X$, by the hypothesis, there exists an open neighborhood U of x such that $\varphi := \pi_x|_U : U \rightarrow \text{tg}_x X$ is injective. Moreover, π_x is open by Lemma 3.1. This implies $\varphi : U \rightarrow \varphi(U)$ is a homeomorphism. Consider the map $\psi := \varphi^{-1} : \varphi(U) \rightarrow U \subset \mathbb{R}^n$. We have $\text{tg} \Gamma_\psi = \text{tg} X|_{\Gamma_\psi}$, which is a continuous trivial bundle. Shrinking U if necessary we may assume that $\text{tg}_y X$ is not orthogonal to $\text{tg}_x X$ for every $y \in U$. The map ψ then satisfies the conditions of Lemma 3.2, so it is of class C^1 , meaning that U is a C^1 manifold. □

Remark 3.5. Theorem 3.4 is slightly stronger than a similar result proved by Gluck (Theorem 10.1, [8]). In the result of Gluck, X is assumed to be a topological manifold instead of a locally closed set as in our statement.

Theorem 3.6 (Two-cones coincidence, Tierno [10]). *A locally closed subset X of \mathbb{R}^n is a C^1 manifold if and only if TX and $\text{ptg}X$ coincide, and both are trivial bundles of vector spaces over X .*

Proof. Since $\text{ptg}X$ is a trivial bundle, it is continuous by Lemma 2.2. On the other hand, $\text{tg}X = \text{ptg}X$, hence $\text{tg}X$ is a continuous trivial bundle.

Let $x \in X$. We denote by $\pi_x : \mathbb{R}^n \rightarrow \text{tg}_x X$ the orthogonal projection. By Theorem 3.4, it suffices to prove that the map π_x is injective on some neighborhood of x in X .

Suppose on contrary that there are sequences of points $\{z_i\}_i$ and $\{z'_i\}_i$ in X converging to x such that $\pi_x(z_i) = \pi_x(z'_i)$ for all i . This implies that $\text{span}(z_i - z'_i)$

accumulates to a line $l \subset \text{tg}_x X^\perp$. By the definition, $l \subset \text{ptg}_x X$. Since $\text{ptg}_x X = \text{tg}_x X$, $l \subset \text{tg}_x X \cap \text{tg}_x X^\perp = 0$, a contradiction. \square

Theorem 3.7 (Definable two-cones coincidence). *A connected, locally closed definable subset of \mathbb{R}^n is a C^1 manifold if and only if $\text{tg}X$ and $\text{ptg}X$ coincide.*

Proof. We just need to show the sufficiency. First we prove that for every $x \in X$, $\text{tg}_x X$ is a linear subspace of \mathbb{R}^n , or equivalently that $\text{tg}X$ is a bundle. Fix $x \in X$, we may identify x with the origin 0. By the hypothesis, $\text{tg}_0 X = \text{ptg}_0 X$ which is symmetric, i.e., if $v \in \text{tg}_0 X$ so is $-v$. It is enough to verify that if $v, w \in \text{tg}_0 X$ then $v + w \in \text{tg}_0 X$. Since $v, -w \in \text{tg}_0 X$ and X is a definable set there exist two curves γ, β in X starting at 0 such that $v \in \text{tg}_0 \gamma$ and $-w \in \text{tg}_0 \beta$ (see Curve selection Lemma). Choose sequences of points $\{x_i\}_i \subset \gamma$ and $\{y_i\}_i \subset \beta$ converging to 0 such that $\|x_i\| = \frac{a}{b}\|y_i\|$, where $a := \|v\|$ and $b := \|w\|$. Thus,

$$v + w = \lim_{i \rightarrow \infty} (a \frac{x_i}{\|x_i\|} - b \frac{y_i}{\|y_i\|}) = \lim_{i \rightarrow \infty} \frac{a}{\|x_i\|} (x_i - y_i) \in \text{ptg}_0 X = \text{tg}_0 X.$$

From now on, we set

$$O_k := \{z \in X : \dim \text{tg}_z X \leq k\}.$$

Fix k and let $z \in O_k$. Take V to be a closed neighborhood of z in X . Since X is locally closed, we may assume V to be a closed set in \mathbb{R}^n so $\text{ptg}X|_V$ is a closed bundle. By Lemma 2.1 (ii), for $y \in V$, the map $y \mapsto \dim \text{ptg}_y X$ is upper-semicontinuous, so is the map $y \mapsto \dim \text{tg}_y X$, meaning that there exists $U \subset V$, an open neighborhood of z , such that $k = \dim \text{ptg}_z X \geq \dim \text{ptg}_y X$ for all y in U . This implies that $U \subset O_k$, hence O_k is an open set in X .

Since X is definable, $\dim \text{tg}_x X \leq \dim X = d$ for every $x \in X$ (see [9], Lemma 1.2), hence $O_d = X$. Set $\mathcal{O} := X \setminus O_{d-1}$, which is a closed set of X .

We denote by X_{Sing} the set of singular points of X , i.e., points at which X fails to be a C^1 manifold of dimension d . Remark that X_{Sing} is a definable set of dimension less than d (see for instance [11], [5]).

Since $O_{d-1} \subset X_{\text{Sing}}$, $\dim O_{d-1} < d$. For $x \in \mathcal{O}$,

$$\text{tg}_x X \supset \text{tg}_x \mathcal{O} \supset \text{tg}_x X \setminus \text{tg}_x O_{d-1}.$$

Taking the closures of all sets above,

$$\text{tg}_x X \supset \text{tg}_x \mathcal{O} \supset \overline{\text{tg}_x X \setminus \text{tg}_x O_{d-1}}.$$

Because $\text{tg}_x X$ is a linear space of dimension d and $\text{tg}_x O_{d-1}$ is a linear subspace of $\text{tg}_x X$ of dimension less than d , $\overline{\text{tg}_x X \setminus \text{tg}_x O_{d-1}} = \text{tg}_x X$. So,

$$\text{tg}_x X = \text{tg}_x \mathcal{O} \subset \text{ptg}_x \mathcal{O} \subset \text{ptg}_x X = \text{tg}_x X.$$

Since \mathcal{O} is closed in the locally closed set X , it is also locally closed. As been shown above, $\text{tg}\mathcal{O} = \text{ptg}\mathcal{O} = \text{ptg}X|_{\mathcal{O}}$ which is a trivial bundle. By Theorem 3.6, \mathcal{O} is a C^1 manifold of dimension d . Next we will prove that $X = \mathcal{O}$, therefore, it is a C^1 manifold.

Let $x \in \mathcal{O}$ and $\pi_x : \mathbb{R}^n \rightarrow \text{tg}_x \mathcal{O}$ be the orthogonal projection. It follows from Theorem 3.4 and Lemma 3.1 that there is an open neighborhood U of x in \mathbb{R}^n such that the restriction of π_x to $U \cap \mathcal{O}$ is injective and $V := \pi_x(U \cap \mathcal{O})$ is an open set in

$\text{tg}_x \mathcal{O}$. Set $W := \pi_x^{-1}(V) \cap U$, so that W is an open neighborhood of x in \mathbb{R}^n with $\pi_x(W) = V$. This implies that

$$\pi_x(W \cap X) = \pi_x(W \cap \mathcal{O}) = V.$$

Since $\text{tg}_x \mathcal{O} = \text{ptg}_x X$, shrinking U if necessary, the restriction of π_x to $W \cap X$ is injective. The sets $W \cap X$ and $W \cap \mathcal{O}$ then are graphs of mappings over the same domain V . On the other hand, $W \cap \mathcal{O} \subset W \cap X$, then $W \cap \mathcal{O} = W \cap X$. This means that $W \cap \mathcal{O}$, an open neighborhood of x in \mathcal{O} , is an open neighborhood of x in X . Thus, \mathcal{O} is an open set in X . Since \mathcal{O} is both closed and open in X and X is connected, \mathcal{O} is equal to X . □

A direct consequence of Theorem 3.6 is:

Corollary 3.8. *Let $X \subset \mathbb{R}^n$ be a locally closed definable set. Suppose that $\text{tg}_x X = \text{ptg}_x X$ for every $x \in X$. Then, each connected component of X is a C^1 manifold.*

Remark 3.9. The definability in the hypothesis of Theorem 3.7 is necessary. Consider the following locally closed sets,

$$X := \overline{\{(x, \sin \frac{1}{x}), x \neq 0\}} \setminus \{(0, 1), (0, -1)\}} \subset \mathbb{R}^2,$$

and

$$Y := \{-1 - \frac{1}{n}, n \in \mathbb{N}\} \cup [-1, 1] \cup \{1 + \frac{1}{n}, n \in \mathbb{N}\}.$$

The sets X and Y are not definable in any o-minimal structure : $X \cap \mathbb{R} \times \{0\}$ and Y have infinitely many component. The set X is connected, $\text{tg} X = \text{ptg} X$, but X is not a C^1 manifold. The set Y has $\text{tg} Y = \text{ptg} Y$, but $[-1, 1]$, a connected component of Y , is not a C^1 manifold.

4. DEFINABLE SETS WITH CONTINUOUS TRIVIAL TANGENT CONES

Let us recall the result proved by Ghomi and Howard [6].

Definition 4.1. Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$. Suppose that the Hausdorff dimension of $\text{tg}_x X$, denoted by $\dim_{\mathcal{H}} \text{tg}_x X$, is an integer k . The lower density $\Theta(X, x)$ of X at the point x is defined as follows: if $x \notin \overline{X}$ then $\Theta(X, x) = 0$, and otherwise,

$$\Theta(X, x) = \liminf_{r \rightarrow 0} \frac{\mathcal{H}^k(X \cap \overline{\mathbf{B}}^n(x, r))}{r^k \mu_k}$$

where \mathcal{H}^k is the k -dimensional Hausdorff measure, μ_k is the volume of the unit ball of dimension k .

Theorem 4.2 (Theorem 1.1, [6]). *Let X be a locally closed subset of \mathbb{R}^n . Suppose that $\text{tg} X$ is a $(n - 1)$ -dimensional continuous trivial bundle. Then,*

- (i) X is a union of C^1 hypersurfaces;
- (ii) if $\Theta(X, x)$ is at most $m < 3/2$ then X is a C^1 hypersurface.

The following example shows that in general the statement (i) of Theorem 4.2 is no longer true when the hyperplanes are replaced by k -planes with $k < n - 1$, meaning that a locally closed subset in \mathbb{R}^n with continuous trivial tangent cone might not be a union of C^1 manifolds.

Example 4.3. We identify \mathbb{C} with \mathbb{R}^2 . Consider the map $h : \mathbb{C} \rightarrow \mathbb{C}$ defined as follows:

$$h(z) := \frac{z^2}{|z|^{\frac{3}{2}}} \text{ if } z \neq 0, \text{ and } h(0) = 0.$$

Denote by $X \subset \mathbb{R}^4$ the graph of h . We have

- (1) X is locally closed;
- (2) $\text{tg}X$ is a 2-dimensional continuous trivial bundle;
- (3) X is not a union of C^1 submanifolds of dimension 2 of \mathbb{R}^4 .

Proof. Remark that h is continuous, hence X is a topological manifold and the condition (1) is automatically satisfied. Moreover, h is smooth except at the origin, where statement (2) is obvious. We now calculate tg_0X . Let $x \in X \setminus \{0\}$, we may write $x = (z, h(z))$ for some $z \in \mathbb{C}^*$. We have

$$\frac{x}{\|x\|} = \frac{(z, h(z))}{\|(z, h(z))\|} = \left(\frac{z}{\sqrt{|z|}} \frac{1}{\sqrt{1+|z|}}, \frac{z^2}{|z|^2} \frac{1}{\sqrt{1+|z|}} \right).$$

Notice that $\frac{z}{\sqrt{|z|}} \frac{1}{\sqrt{1+|z|}} \rightarrow 0$ when $z \rightarrow 0$. Hence $\text{tg}_0X \subset \{z = 0\}$. On the other hand, if $z = re^{i\theta}$ and $r \rightarrow 0$, $\frac{z^2}{|z|^2} \frac{1}{\sqrt{1+|z|}} \rightarrow e^{2i\theta}$. Thus, $(0, e^{2i\theta}) \in \text{tg}_0X$ for all θ . This implies that $\text{tg}_0X = \{z = 0\}$.

Write $z = z_1 + iz_2$ and $h = h_1 + ih_2$. For $x = (z, h(z))$, $z \neq 0$, tg_xX is generated by two vectors $u = (1, 0, \frac{\partial h_1}{\partial z_1}, \frac{\partial h_2}{\partial z_1})$ and $v = (0, 1, \frac{\partial h_1}{\partial z_2}, \frac{\partial h_2}{\partial z_2})$. Denote by ∂_1 and ∂_2 the directional derivatives in the variable z along z_1 -axis and z_2 -axis respectively. We know that

$$\begin{aligned} \partial_1 h &= \frac{\partial h_1}{\partial z_1} + i \frac{\partial h_2}{\partial z_1} \\ \partial_2 h &= -i \frac{\partial h_1}{\partial z_2} + \frac{\partial h_2}{\partial z_2} \end{aligned}$$

Note that $h = \frac{z^2}{(z\bar{z})^{\frac{3}{4}}}$. Computation gives,

$$\begin{aligned} \partial_1 h &= \frac{5}{4} \frac{z}{|z|^{\frac{3}{2}}} \partial_1 z - \frac{3}{4} \frac{z^3}{|z|^{\frac{7}{2}}} \partial_1 \bar{z} = \frac{5}{4} \frac{z}{|z|^{\frac{3}{2}}} - \frac{3}{4} \frac{z^3}{|z|^{\frac{7}{2}}} \\ \partial_2 h &= \frac{5}{4} \frac{z}{|z|^{\frac{3}{2}}} \partial_2 z - \frac{3}{4} \frac{z^3}{|z|^{\frac{7}{2}}} \partial_2 \bar{z} = \frac{5}{4} \frac{z}{|z|^{\frac{3}{2}}} + \frac{3}{4} \frac{z^3}{|z|^{\frac{7}{2}}}. \end{aligned}$$

If z tends to 0, then $|\partial_1 h|$ and $|\partial_2 h|$ tend to ∞ . Therefore,

$$\begin{aligned} \|u\| &= (1 + |\partial_1 h|^2)^{\frac{1}{2}} \rightarrow \infty, \text{ and} \\ \|v\| &= (1 + |\partial_2 h|^2)^{\frac{1}{2}} \rightarrow \infty. \end{aligned}$$

Hence $\lim_{x \rightarrow 0} \text{tg}_xX = \text{tg}_0X$. This implies (2).

Now we show (3). Denote by $\pi_0 : \mathbb{R}^4 \rightarrow \text{tg}_0X$ the orthogonal projection from \mathbb{R}^4 onto tg_0X :

$$\pi_0|_X : X \ni (z, h(z)) \mapsto h(z) \in \text{tg}_0X.$$

This map is not injective in any neighborhood of 0 since $\forall z, h(z) = h(-z)$. By Theorem 3.4, X is not a C^1 manifold. Since X is a connected topological manifold, it cannot be the union of two or more C^1 manifolds of dimension 2. \square

Definition 4.4 ([3],[4],[9]). Let $X \subset \mathbb{R}^n$ be a definable set and let $x \in \mathbb{R}^n$. Suppose that $\dim X = k$. It is known that the following limit always exists

$$\theta(X, x) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^k(X \cap \overline{\mathbf{B}}^n(x, r))}{r^k \mu_k}.$$

We call it the *density* of X at the point x .

Remark 4.5. The notions of lower density and density are not the same even for definable sets. For example, consider $X := \{(x, y, z) \in \mathbb{R}^3 : z^4 = x^2 + y^2\}$. It is easy to see that $\Theta(X, 0) = \infty$ while $\theta(X, 0) = 0$. However, if X is a definable set and $\text{tg}X$ is a trivial bundle then $\dim_{\mathcal{H}} \text{tg}_x X = \dim \text{tg}_x X = \dim X$, and therefore $\Theta(X, x) = \theta(X, x)$ for every $x \in X$.

Lemma 4.6. (i) Let X, Y be definable sets of the same dimension. If $\dim(X \cap Y) < \dim X$, then $\theta(X \cup Y, x) = \theta(X, x) + \theta(Y, x)$. If $X \subset Y$, then $\theta(X, x) \leq \theta(Y, x)$.
(ii) If X is a definable set then $\theta(X, x) \geq \theta(\text{tg}_x X, 0)$.

Proof. (i) is a direct consequence of the definition of density, and (ii) of Theorem 3.8 of [9]. \square

Theorem 4.7. Let X be a locally closed definable subset of \mathbb{R}^n . If $\text{tg}X$ is a continuous trivial bundle and $\theta(X, x) < 3/2$ for every $x \in X$, then X is a C^1 manifold.

Remark 4.8. The condition on the density above is sharp, since

$$X := \{(x, y); y = 0\} \cup \{(x, y); x > 0, y = x^2\}$$

satisfies all other hypothesis of Theorem 4.7 and $\theta(X, 0) = 3/2$.

Proof. Denote by $\pi_x : \mathbb{R}^n \rightarrow \text{tg}_x X$ the orthogonal projection. Suppose on the contrary that X is not a C^1 manifold. By Theorem 3.4, there exists $x \in X$ such that there is no neighborhood of x in X to which the restriction of π_x is injective. We may assume that x coincides with the origin 0 and $\text{tg}_x X = \mathbb{R}^k$ where $k = \dim X$. The map π_x now becomes $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$, the orthogonal projection to the first k coordinates.

By Lemma 3.1, there is an open neighborhood U of 0 in X such that $\pi|_U$ is an open map. Hence there exists $r > 0$ such that $\mathbf{B}^k(0, r) \subset \pi(U)$. Shrinking U if necessary we assume that $\pi(U) = \mathbf{B}^k(0, r)$. We also assume that for every $x \in U$, $\text{tg}_x X$ is not orthogonal to \mathbb{R}^k .

By the uniform finiteness on fibres of definable sets, there exists $N \in \mathbb{N}$ such that for every $z \in \mathbf{B}^k(0, r)$, $|\pi^{-1}(z) \cap U|$, the number of connected components of $\pi^{-1}(z) \cap U$, does not exceed N . In fact, in this case, $|\pi^{-1}(z) \cap U| = \text{card}(\pi^{-1}(z) \cap U)$ where card denotes the cardinality. If otherwise, there is a connected component of $\pi^{-1}(z) \cap U$, write F , such that $\dim F \geq 1$. Since F is definable, there is a point $\tilde{z} \in F$ such that $\dim \text{tg}_{\tilde{z}} F = \dim F \geq 1$. But $F \subset (\{\tilde{z}\} \times \mathbb{R}^{n-k})$, so $\text{tg}_{\tilde{z}} F \subset (\{0\} \times \mathbb{R}^{n-k})$. This implies that $\text{tg}_{\tilde{z}} F \subset \text{tg}_{\tilde{z}} X \cap \{0\} \times \mathbb{R}^{n-k}$. Since $\tilde{z} \in U$, $\text{tg}_{\tilde{z}} X$ is not orthogonal to \mathbb{R}^k , then $\text{tg}_{\tilde{z}} X \cap (\{0\} \times \mathbb{R}^{n-k}) = \{0\}$, which gives a contradiction.

Set

$$S_\kappa := \{z \in \mathbf{B}^k(0, r) : \text{card}(\pi^{-1}(z) \cap U) = \kappa\}.$$

Then $\{S_\kappa\}_{\kappa=1}^N$ becomes a definable partition of $\mathbf{B}^k(0, r)$. We may assume that $0 \in \overline{S_\kappa}, \forall \kappa$ and $S_N \neq \emptyset$. Note that $N \geq 2$ since the restriction of π is not injective on any neighborhood of 0. We claim that

- (a) S_N is an open set,
- (b) $|(\pi^{-1}(S_N) \cap U)| = N$,
- (c) Each connected component of $\pi^{-1}(S_N) \cap U$ is a C^1 manifold.

We now give a proof of the claim.

Let $q \in S_N$. Since $\text{card}(\pi^{-1}(q) \cap U) = N$, we may write $\pi^{-1}(q) \cap U = \{q_1, \dots, q_N\}$. There is $\epsilon > 0$ sufficiently small such that $K_i \cap K_j = \emptyset$ for $i \neq j$, where $K_i := \mathbf{B}^n(q_i, \epsilon) \cap U, i \in \{1, \dots, N\}$. Since the map $\pi|_U$ is open, there exists an open neighborhood V_q of q in \mathbb{R}^k such that $V_q \subset \pi(K_i), \forall i \in \{1, \dots, N\}$. For $q' \in V_q, \forall i \in \{1, \dots, N\}, \pi^{-1}(q') \cap K_i \neq \emptyset$, hence $\text{card}(\pi^{-1}(q') \cap U) \geq N$. By the definition of N , $\text{card}(\pi^{-1}(q') \cap U)$ cannot exceed N , therefore $\text{card}(\pi^{-1}(q') \cap U) = N$, meaning $q' \in S_N$. This implies $V_q \subset S_N$, or equivalently S_N is open, (a) is proved.

Denote by A_1, \dots, A_m the connected components of $\pi^{-1}(S_N) \cap U$. Note that for every i , A_i is an open set in U , so $\pi(A_i)$ is an open set in \mathbb{R}^k since $\pi|_U$ is an open map. Assume that (b) does not hold, i.e., $m > N$. Then, there exists an $i \leq m$ such that $\pi(A_i)$ does not cover the whole of S_N , for simplicity we assume $i = 1$. Since $\pi(A_1) \subsetneq S_N$ is open, there exists $p \in S_N \cap \partial\pi(A_1)$. Writing $\pi^{-1}(p) \cap U = \{p_1, \dots, p_N\}$, there is an $\alpha \in \{1, \dots, N\}$ such that p_α belongs to $\overline{A_1}$. But $p_\alpha \notin A_1$ since $\pi(p_\alpha) \in \partial\pi(A_1)$, then $p_\alpha \in A_\beta, \beta \neq 1$. This implies that $\overline{A_1} \cap A_\beta \neq \emptyset$, so A_1 and A_β are the same connected component, which is a contradiction.

It follows from (b) that for each $i \in \{1, \dots, N\}$ the restriction $\pi|_{A_i} : A_i \rightarrow S_N$ is a bijection. In other words, $A_i = \Gamma_{\xi_i}$ with $\xi_i : S_N \rightarrow \mathbb{R}^n, \xi_i(y) = \pi^{-1}(y) \cap A_i$. Note that $\text{tg} A_i = \text{tg} X|_{A_i}$ which is a continuous trivial bundle, its fibers are, moreover, not orthogonal to \mathbb{R}^k by the construction. This shows that the function ξ_i satisfies conditions of Lemma 3.2, hence Γ_{ξ_i} is a C^1 manifold. This ends the proof of (c).

Let $z \in S_N$ such that $\|z\| < r/4$. Let z' be a point realizing the distance from z to the boundary ∂S_N of S_N . Since $0 \in \partial S_N, s := \|z' - z\| \leq r/4$, and then $\|z'\| \leq \|z' - z\| + \|z\| \leq r/4 + r/4 = r/2$, hence $z' \in \pi(U) = \mathbf{B}^k(0, r)$. Since $\mathbf{B}^k(z, s) \subset S_N$, $\pi^{-1}(\mathbf{B}^k(z, s)) \cap U$ has exactly N connected components, denoted by $\{B_1, \dots, B_N\}$. Remark that $z' \notin S_N$ (i.e., $\text{card}(\pi^{-1}(z') \cap U) < N$) but $z' \in \pi(\overline{B_i}), \forall i \in \{1, \dots, N\}$, so there are $i, j, i \neq j$ such that $\overline{B_i} \cap \overline{B_j} \cap \pi^{-1}(z') \neq \emptyset$. Take $w \in \overline{B_i} \cap \overline{B_j} \cap \pi^{-1}(z')$. Take $C \subset \mathbf{B}^k(0, r)$ a small closed ball outside $\overline{\mathbf{B}^k}(z, s)$ and tangent to $\overline{\mathbf{B}^k}(z, s)$ at z' . Denote by D the connected component of $\pi^{-1}(C) \cap U$ which contains w .

Since $\text{tg}_w U$ is a k -dimensional linear subspace of \mathbb{R}^n and $\pi|_{\text{tg}_w U} : \text{tg}_w U \rightarrow \mathbb{R}^k$ is a linear bijective map, D, B_i, B_j are disjoint definable sets of dimension k in U and $\text{tg}_w D, \text{tg}_w B_i, \text{tg}_w B_j$ are half k -planes. By Lemma 4.6,

$$\begin{aligned} \theta(X, w) &= \theta(U, w) \geq \theta(B_i, w) + \theta(B_j, w) + \theta(D, w) \\ &\geq \theta(\text{tg}_w B_i, 0) + \theta(\text{tg}_w B_j, 0) + \theta(\text{tg}_w D, 0) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}. \end{aligned}$$

This contradicts the hypothesis that $\theta(X, x) < \frac{3}{2}$ for every $x \in X$. \square

Acknowledgements. This research has been supported by ANR project STAAVF. The third author has been also received the funding from NCN grant 2014/13/B/ST1/00543.

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